1

Relations

This book starts with one of its most abstract topics, so don't let the abstract nature deter you. *Relations* are quite simple but like virtually all simple mathematical concepts they have their subtle aspects. Relations are defined on collections (or *sets*) of objects. The objects are usually numbers but can be anything. For example our collection of objects could be {1, 2, 3, a, c, ☞, ♥}. Also the collections could be infinite like the set of natural numbers: {1, 2, 3, ... }. In this book we will almost always use finite collections.

By a *relation*, we mean a collection of ordered pairs from the collection of objects. For example, on the collection of objects {1, 2, 3, a, c, ☞, ♥}, we can define the relation {(1,2) (2,a) (☞,2) (2,☞)}. The pairs are said to be *ordered* because we consider (1,2) and (2,1) to be different pairs. Since we are only dealing with ordered pairs instead of, say, ordered triples we are actually using *binary* relations.

Why Do We Use the Word *Relation*?

Consider the collection of objects {Bob, Ted, Carol, and Alice}. The relation *marriage* might be defined as {(Bob, Alice) (Alice, Bob) (Ted, Carol) (Carol, Ted)}. Here each pair represents a married couple. We have both (Bob, Alice) and (Alice, Bob) because Bob is married to Alice and Alice is married to Bob. The relation *older* might be defined as {(Bob, Ted) (Bob, Alice) (Bob, Carol) (Alice, Ted) (Alice, Carol) (Carol, Ted)}. (In each case the older person is on the left.) In this instance Bob is the eldest person, Alice is the next oldest, then Carol, and Ted is the youngest. Note that the ordered pairs can be written in any order.

Given the collection of numbers {1, 2, 3, π) we can define the relation *larger* as {(π, 3) (π, 2)(π, 1) (3, 2) (3, 1) (2, 1)}. The relation *equality* is given by {(π,π) (3, 3) (2, 2) (1, 1)}. The relation *greater than or equal to* is given by {(π, 3) (π, 2)(π, 1) (3, 2) (3, 1) (2, 1) (π,π) (3, 3) (2, 2) (1, 1)}.

Functions

Functions are a type of relation of great importance throughout mathematics. In this book we will not deal with the function concept; functions are given here only as an example.[1] A relation is a function if and only if no element occurs as a first element in more than one ordered pairs. Suppose for example that we have a relation, $\mathbb{R}$, defined on the natural numbers (positive integers). $\mathbb{R}$ then is a collection of ordered pairs of natural numbers. If $\mathbb{R}$ is a function and (3,9) belongs to $\mathbb{R}$ then there is no other ordered pair beginning with 3. However, there can be other ordered pairs ending with 9. A good way of thinking of functions is with each pair consisting of an input and an output. The second element is the output corresponding to the first element. In particular we say that *the second element is a function of the first element*. Note that in the example above where the relation is marriage the relation is also a function. However, in a society that is not monogamous the marriage relation is not a function.

Permutations

Another type of relation (which also happens to be a function) is a *permutation*. A permutation can be thought of informally as a rearrangement. For example, the purpose of shuffling cards is to *permute* the deck. In a permutation (of a finite set) each element occurs once and only once as a first element and once and only once as a second element.[2] Given the set {1, 2, 3, 4} the relation {(1,2) (2,3) (3,1) (4,4)} is a permutation. A perfectly good way of thinking of permutations are like the game musical chairs. In this particular example 1 moved to 2's seat; 2 moved to 3's seat; 3 moved to 1's seat, and 4 sat back in his old seat.

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[1]We can use various functions without making explicit use of the function concept.

[2]This definition is only valid for permutations on finite sets. For the more advanced readers, we can say that a permutation of a non-empty (possibly infinite) set is a one-to-one mapping from the set onto itself.

Some New Notation

Instead of writing that (1,2), (2,1) and (2,3) belong to the relation ℝ it is easier to write that 1~2 and 2~1 and that 2~3, with some symbol like ~ which is often specific to that relation. This is the way that relations are usually written. For example, we do not say that (2,2) belongs to the equality relation; instead we say 2 = 2.

Equivalence Relations

A particularly important type of relation which we will use later *are equivalence relations*. An equivalence relation has three properties:

- For any object, say A, A~A. (An object is equivalent to itself).
- If A~B, then B~A. (If A is equivalent to B, B is equivalent to A.)
- If A~B and B~C then A~C.
  (If A is equivalent to B and B is equivalent to C, then A is equivalent to C.)

These properties are respectively known as *reflexivity*, *symmetry*, and *transitivity*.[1] For example, given the objects {1, 2, 3, 4} if ℝ is an equivalence relation on these objects, then ℝ must contain at least the pairs {(1,1) (2,2) (3,3) (4,4)} because of the reflexivity property. If it contains the pair (3,2) it must contain the pair (2,3) because of the symmetry property. If ℝ contains the pairs (2,3) and (3,1) it must contain the pair (2,1) because of the transitivity property.

The prototype equivalence relation is equality itself. Equality always satisfies the three rules given above. However, if we look at the relations < (less than) and ≤ (less than or equal to) defined on the integers, they violate the property of symmetry.

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[1] I have to give the names of these properties or mathematicians would shun me.

Partitions

If we have an equivalence relation then we have a *partition* (and vice versa). A partition is a collection of classes of objects, such that every object belongs to one and only one class. (A room divider partitions the objects of its room.) If ~ is an equivalence relation and A~B, we say that A and B belong to the same equivalence class, or the same partition class (they are the same thing). Similarly, if we have a partition of objects, we define the equivalence relation A~B whenever A and B belong to the same partition class.

Example Hopefully an example will make this much clearer. **Figure 1** is a partition with four classes represented by four boxes. What makes it a partition is each element is in a class and no element is in two classes. Now we will deduce the equivalence relation implied by this partition. We say that two elements are related if and only if they belong to the same class. This means that each element is related to itself; for example a~a. This is the property of reflexivity. In this example a~d, but we also have d~a; that is the property of symmetry. We do not have a~b since a and b are in different partition classes. It is not hard to see that in a partition that the property of transitivity will also hold. For example, b~g and g~f imply b~f.

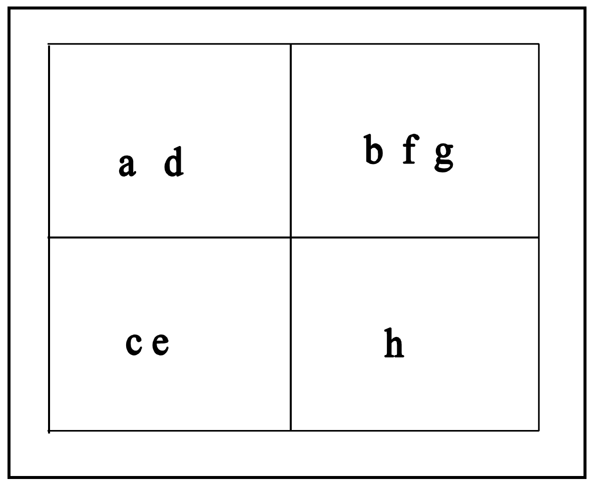

Figure 1 A Partition With Eight Elements and Four Classes

It is important to understand that if we have an equivalence relation ~ (on some set of objects) that we can partition the objects by that relation. We simply put two objects x and y in the same class if and only if $x \sim y$. It is thus true that **there is an equivalence between partitions and equivalence relations**.

The topics discussed so far are abstract. It is not required or expected that you master these concepts. However, if you are a mathematics major or any serious student of the mathematical sciences you will have to master these concepts at some time.