18

Matrices

Definitions and Basic Operations

Matrix algebra is also known as *linear algebra* and it is one of the most important disciplines of mathematics. It pervades virtually all of mathematics, but it could be argued to be closest to geometry. In this chapter, we will look at the relation between matrices and graphs.

$$(3) \qquad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3.5 & 7 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -1 \\ 5 \\ 0 \end{pmatrix}$$

A 1 by 1 Matrix, a 2 by 3 Matrix and a 4 by 1 Matrix.

A matrix is a rectangular array of numbers. An (m, n) matrix is an array with m rows and n columns and is said to be *m by n*.[1] We will refer to the element in the m'th row and the n'th column as the *{m,n} element*. For example, in the (2,3) matrix in the above box, the {1,2} element is -1. The {2,3} element is 7. We will be interested only in the operation of multiplication of matrices. However, in this section we will look at a couple of more elementary operations that are included for completeness. These two operations are matrix addition and multiplication of a matrix by a scalar.

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[1]The notation (m, n) here refers to a matrix of size m by n when previously the same notation referred the greatest common divisor of m and n. Fortunately the context usually makes it clear which applies.

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 0 & 1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \\ -4 & 2 \end{pmatrix}$$

The Sum of Two 3 by 2 Matrices is a 3 by 2 Matrix

Two matrices can be added if and only if they have the same dimensions. That is, each must have the same number of rows and columns. In that case, their sum carried out element by element. The new matrix is also of size (m,n) and its {i,j} element is the sum of the respective {i,j} elements.

$$2 \cdot \begin{pmatrix} -3 & 1 \\ -1 & 2.5 \\ -4 & \pi \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ -2 & 5 \\ -8 & 2\pi \end{pmatrix}$$

Multiplication of a Matrix by the Scalar 2.

A *scalar*, in most applications, means a real number. Scalar multiplication of a matrix means multiplying a real number times a matrix. The scalar can be multiplied times the matrix from either the left or right. The operation consists of multiplying each element of the matrix by the scalar.

Multiplication of Matrices

Multiplication of one matrix by another, is by far the most important matrix operation and will be used in much of the rest of this book. Two matrices can be multiplied if the left matrix has the same number of columns as the right matrix has rows. Just look at their dimensions. An (m,n) matrix can be multiplied times an (n,p) matrix and their product has dimension (m,p). That is the product has the same number of rows as the left matrix and the same number of columns as the right matrix.

Already without any idea of how to do the product of two matrices, we can show that given two matrices A and B, it is not necessarily true that A·B must equal B·A. For example if A has dimension (1,4) and B has dimension (4,1) then A·B is of dimension (1,1) and B·A has dimension (4,4), so in this case A·B can't equal B·A. If A has dimension (1,4) and B has dimension (4,2) then A·B has dimension (1,2) but B·A does not even exist.

To summarize, we now know when we can multiply two matrices and we know the dimension of the product matrix. We now just have to learn how to do it. **The secret of matrix multiplication is that it is a generalization of multiplying a row matrix times a column matrix.** Incidently, it is customary to refer to a row matrix or column matrix as a *vector*.[1] A row matrix is of form (1,n) and a column matrix is of form (n,1); so their product must be of dimension (1,1). (If we multiplied them in the opposite order, we would have a product of dimension (n,n).) We know that we can multiply a (1,4) vector times a (4,1) vector, but we cannot multiply a (1,4) vector times either a (5,1) vector or a (3,1) vector. That is, our requirement for multiplying a row times a column is that the two vectors have the same number of elements. The multiplication is carried out by multiplying the corresponding elements of each vector (that is the first times the first, and the second times the second, etc) and then by adding the products; "matching" elements are multiplied and then their products are summed. This is easier to learn by example than by words. For example: $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix} = (24)$, because

$1 \cdot 5 + 3 \cdot 7 + (-2) \cdot 1 = 24$.

If we multiply a (m,n) matrix times an (n,p) matrix, we get a (m,p) matrix. The {i,j} element of the product is just the i'th row of the first matrix times the j'th column of the second matrix. For example, given the (2,3) and (3,1) matrices:

[1] Again this is purely custom. Vectors often are defined abstractly in terms of vector spaces. In that case vectors have certain properties and might not be column or row matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

A·B is a (2,1) matrix. Its {1,1} element is the first row of A times the first column of B (that B only has one column simplifies things a little). Its {2,1} element is the second row of A times the first column of B. We have: $\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$

because $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} = -9, \quad \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} = -1$.

As we have shown above, matrices are not commutative, that is A·B ≠ B·A (usually), but we do have associativity: A(BC) = (AB)C, and we also have distribution: A(B + C) = AB + AC and (B + C)A = BA + CA. There is no trick to proving that matrix multiplication is associative and distributes over addition. These things are proven by applying the definition of multiplication to the cases and showing that the results are equal. The proofs are a little laborious, but they are not difficult.

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 1 \\ 6 & -2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 3 & -2 & -6 & 1 \\ -2 & 4 & -1 & 5 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

☐ **Exercise 1** For the matrices in the box above, perform the following multiplications: A·B, B·C, C·D, A·C, D·B.

To multiply matrices A (m, n) and B (n, p) we need to set aside (m, p) space for the product matrix C. Also, we need to initialize every entry of C to 0. Having done that, the multiplication algorithm is:

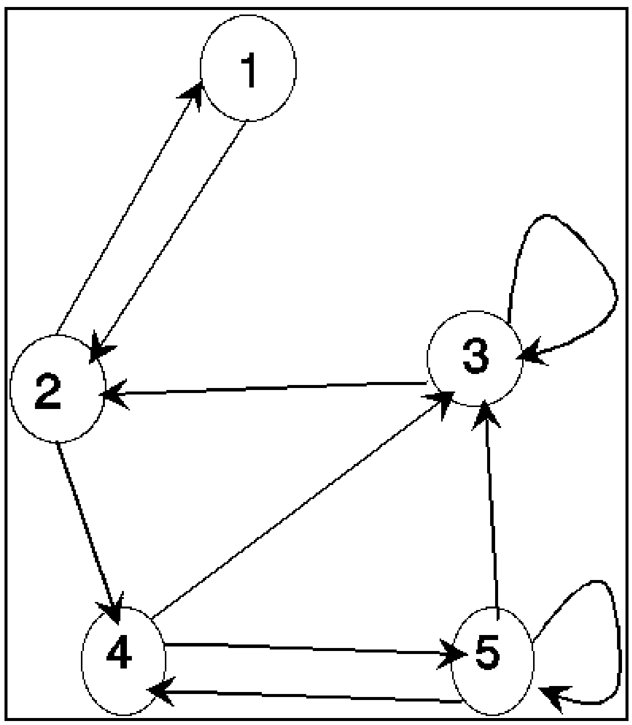
```
For Row ← 1 to m
    For Col ← 1 to p
        For k ← 1 to n
        C[Row, Col] ← C[Row, Col] + A[Row, k]·B[k, Col]
        Next k
    Next Col
Next Row.
```

The Multiplication Algorithm for Matrices.

Applications of Matrices to Graphs

Incidence Matrices

There are many ways to represent graphs by matrices. We will concentrate on *incidence matrices*. We represent an m-vertex graph by a square matrix of dimension (m,m). The {i,j} entry of the matrix has an entry of 1 if there is an arc from vertex i to vertex j. Otherwise {i,j} has a 0. For example, consider the graph:



Its matrix is:

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

This means that to represent an m-vertex graph on a computer you can use an m by m sized array M, with M(i,j) = 1 if there is an arc from vertex i to vertex j. (Sometimes graphs have more than one arc from a given vertex to another. In that case this representation does not work.)

Powers of Incidence Matrices

One of the most interesting features of incidence matrices is the meaning of their powers. The second power of G is $G^2 = G \cdot G$, and the n'th power, G^n is the product of G times itself n times. We can illustrate the significance of these powers by looking at the second and third powers of the above matrix, G. We have:

$$G^2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 1 & 2 \end{pmatrix} \qquad G^3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 3 & 3 \end{pmatrix}$$

These matrices do not satisfy the definition of incidence matrices since they do not just have 0's and 1's, but they are highly related to incidence matrices. Consider the matrix G^2. The {4,3} entry which is 2 indicates that there are 2 ways to go from vertex 4 to vertex 3 in the graph via paths of exactly 2 arcs. If we look at the vertex {5,3} in matrix G^3, its 6 implies there are 6 paths of length 3 from vertex 5 to 3. They are: 5-5-5-3; 5-5-3-3; 5-3-3-3; 5-5-4-3; 5-4-3-3; 5-4-5-3. In general if n is a positive integer, G^n gives the number of paths of exactly n arcs from each vertex to each other vertex. Note from the last example that these paths can be highly redundant in that one path might go through the same vertex more than once.

Permutation Matrices

Remember a graph is a permutation of its vertices, if there is exactly one arc leaving each vertex and one arc entering each vertex. It follows that the incidence matrix M represents a permutation graph, H, if and only if M is a square matrix containing only 0's and 1's and there is exactly one 1 in each row and each column.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Figure 1 All 3-by-3 Permutation Matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}3\\4\\5\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}4\\5\\3\end{pmatrix}\quad\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}5\\3\\4\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}3\\5\\4\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}4\\3\\5\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}5\\4\\3\end{pmatrix}$$

Figure 2 Multiplication of a Column Matrix by Permutation Matrices

Remember also that one interpretation of a permutation is that it is a rearrangement of its elements. Now, if we perform a permutation of a collection of objects and then we perform another permutation of those same objects, the net result is a permutation of the objects. Similarly, if the matrix M represents the first permutation and the matrix N represents the second permutation, the matrix P = M·N represents the net permutation. In **Figure 2** are all of the permutation matrices of size (3,3). If you multiply any two of these matrices, you will get one of the six matrices as the product. In **Figure 2** each 3-by-3 permutation matrix is multiplied times the same column vector (matrix) $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. Each product is a different permutation of the three entries of the vector.

☐ **Exercise 2** Construct two permutation matrices, M, N of the same size (so that they can be permutations of the same set of objects) and so that $M \cdot N \neq N \cdot M$. The existence of such matrices means that permutations do not commute. That is, if I perform two permutations on a collection of objects, the net result is (usually) dependent on the order that I perform the permutations.

☐ **Exercise 3** Find two permutations that do commute ($M \cdot N = N \cdot M$). Hint: Consider permutations that affect disjoint collections of objects. Remember, a permutation may leave particular objects unmoved.

Periodic Matrices (Feel free to skip and go to the next topic: Inverses)

Periodic Markov chains will be very important later and we will redefine them and prove other properties in Section 30. A graph is periodic if it is strongly connected (it is possible to go from any vertex to any other with the direction of the arcs) and if on leaving any vertex we only return on multiples of some integer greater than 1. (Note we are only discussing finite graphs!) The lowest number which satisfies this property is said to be the *period* of the graph. Note the least period a graph can have is 2. Also, this definition assumes a very important property that will be proven in the Appendix to Section 30. Suppose a graph is strongly connected. If on leaving a given node we can only return on multiples of 6, and 6 is the smallest integer (greater than one) with this property, then the property is true of all vertices of the graph. That is the graph is of period 6 because each node is of period 6 (this will all be made precise in Section 30). For example the graph in **Figure 3** node d looks to be of period 4, but it is in fact of period 2 just like node a is. On leaving d, you might return after 6 transitions, and of course 6 is not a multiple of 2.

Suppose that M is a matrix of a strongly connected graph. It will be shown in the Appendix to this section that M has 0's in all of its powers if and only if the graph is periodic. Furthermore, (it will be shown) that if a graph (with matrix M) is strongly connected and is not periodic, then there is a power n, such that M^k contains no 0's for $k \geq n$.

Some Examples

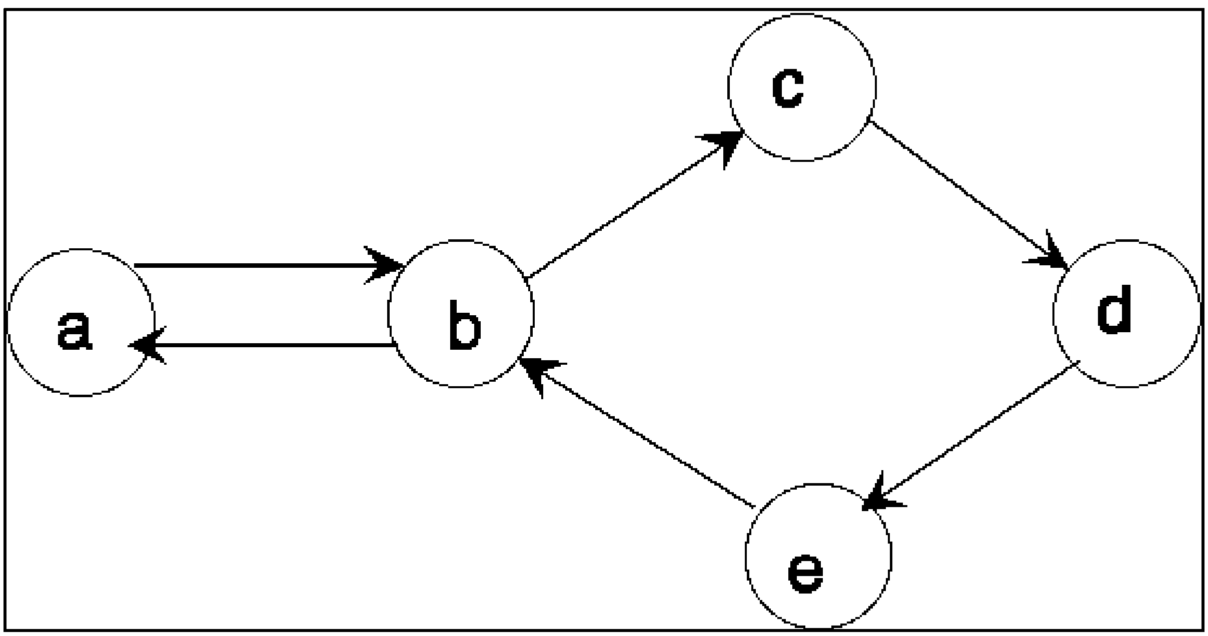


Figure 3 A Graph of Period 2 With Interesting Incidence Matrices

☐ **Exercise 4** Construct the incidence matrix of the graph in **Figure 3**. Compute its first six powers.

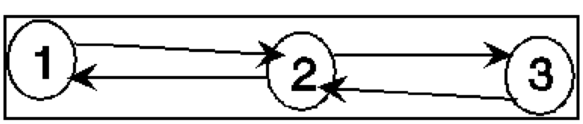


Figure 4 A Graph of Period 2

The graph in **Figure 4** has matrices M, M^2 and M^3 as follows:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Since the graph is periodic of period 2, all powers of its matrix will have 0's. In fact all even powers will be alike and all odd powers will be alike as to the position of the 0's.

If we consider the graph in **Figure 5**, its matrix is $M = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and all of its powers have 0's. However, the graph is not periodic. It can have both of these properties because it is not strongly connected.

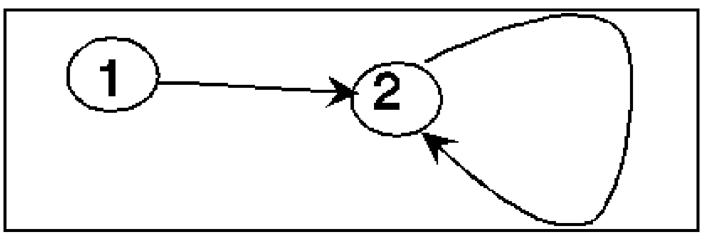


Figure 5 A Graph Not Strongly Connected

Inverses

Identity Matrices

$$\begin{bmatrix} 1 & 3 & 5 & 1 \\ -2 & 3 & 1 & 6 \\ -5 & 6 & 7 & -2 \\ 9 & -5 & 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 1 \\ -2 & 3 & 1 & 6 \\ -5 & 6 & 7 & -2 \\ 9 & -5 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 5 & 1 \\ -2 & 3 & 1 & 6 \\ -5 & 6 & 7 & -2 \\ 9 & -5 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 1 \\ -2 & 3 & 1 & 6 \\ -5 & 6 & 7 & -2 \\ 9 & -5 & 6 & 6 \end{bmatrix}$$

Let us consider square matrices of dimension (n,n). Such matrices can always be multiplied times each other in any order. The matrix which has 0's everywhere except for the main diagonal which is all 1's is known as the *identity matrix* and is denoted by I_n. Usually the dimension is clear and the n is left off. I_n has the property that for any other square matrix M of the same dimension that I·M = M·I. I_n is the special permutation matrix that sends each element back to itself. If for some matrix M there is a matrix M' such that M·M' = M'·M = I then M' is called the *inverse* of M.

☐ **Exercise 5** Show that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse. Hint: construct a (2,2) matrix whose elements are variables, and try to solve the implied equation.

It will be useful in Section 35 to find inverses of matrices. Since we are studying matrices in this chapter, we will learn to do inverses here. The following technique is not only the best known and perhaps the easiest, but is good from a computational standpoint.[1] Anytime a matrix has an inverse, this method will find it. We take the (square) matrix in question and place next to it on the right an identity matrix of the same dimension. We operate on the first matrix until we have turned it into an identity matrix. Simultaneously, everything we do to the first matrix we do to the second. When we are finished, the matrix on the right is the inverse to the matrix we started with on the left. The legal operations are the following:

I. It is permissible to exchange any two rows.

II. A row can be multiplied by a non-zero constant.

III. Any multiple of a row can be added to any other row.

If it is impossible to turn a matrix into the identity (of its dimension) then it has no inverse.

[1]That this technique is completely reliable is not hard to prove. However, the machinery for proving it is not otherwise relevant to this book. The technique is covered in any book on linear algebra or matrix theory.

The general strategy of inverting a matrix is to take care of the columns from left to right. As an example, let us find the inverse of the incidence matrix that we studied earlier:

Original matrix on left, identity on right.

0	1	0	0	0	1	0	0	0	0
1	0	0	1	0	0	1	0	0	0
0	1	1	0	0	0	0	1	0	0
0	0	1	0	1	0	0	0	1	0
0	0	1	1	1	0	0	0	0	1

Interchange first two rows.

1	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	0
0	1	1	0	0	0	0	1	0	0
0	0	1	0	1	0	0	0	1	0
0	0	1	1	1	0	0	0	0	1

Subtract second row from third.

1	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	-1	0	1	0	0
0	0	1	0	1	0	0	0	1	0
0	0	1	1	1	0	0	0	0	1

Subtract third row from fourth and fifth.

1	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	-1	0	1	0	0
0	0	0	0	1	1	0	-1	1	0
0	0	0	1	1	1	0	-1	0	1

Interchange fourth and fifth rows.

1	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	-1	0	1	0	0
0	0	0	1	1	1	0	-1	0	1
0	0	0	0	1	1	0	-1	1	0

Subtract fourth row from first row.

1	0	0	0	-1	-1	1	1	0	-1
0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	-1	0	1	0	0
0	0	0	1	1	1	0	-1	0	1
0	0	0	0	1	1	0	-1	1	0

Subtract fifth row from fourth row. Add fifth row to first row

1	0	0	0	0	0	1	0	1	-1
0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	-1	0	1	0	0
0	0	0	1	0	0	0	0	-1	1
0	0	0	0	1	1	0	-1	1	0

We have just shown that: $$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Almost always if a matrix is invertible and has only integer entries then its inverse with have non-integer entries (unlike the example just done).

☐ **Exercise 6** Invert the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(Note: That M' is often denoted by M^{-1}).

Appendix to Section 18

Proofs

We assume that all graphs are finite.

Theorem: If G is an incidence matrix, then G^n is the matrix enumerating the number of paths of length n.

Proof: This is proven by induction. Clearly, the statement is true if n = 1. We then need only show that if the statement is true for any positive integer n, that it is true for n + 1. Writing $G^{n+1} = G \cdot G^n$ the {i,j} entry of G^{n+1} is the sum of all of the products {i,k}·{k,j} as k ranges over all of the vertices of the graph. {i,k} is a 1 or a 0 depending on whether there is an arc from vertex i to vertex k, and {k,j} is the number of paths of length n from vertex k to vertex j. The product {i,k}·{k,j} is then the number of paths from vertex i to vertex j with k as the second vertex. Since all of the possible second vertices are considered in the sum, the {i,j} entry of G^{n+1} is the number of paths of length n + 1 from vertex i to vertex j.

For the following proofs we will use the fact that every node of a strongly connected periodic graph is periodic (and has the same period). This fact is proven in the Appendix to Section 30. Since that proof does not rely on any of the material in this appendix, there is no problem of circular logic.

Theorem: If a matrix represents a strongly connected graph, all powers of its matrix contain 0's if and only if the graph is periodic.

Proof: We will assume the matrix M is of an n-element graph, G, and is thus n-by-n. Suppose G is periodic and is of period m. (Remember $m > 1$.) Let x be any node of the graph. Then the {x,x} entry of M^h ($h \geq 1$) must be 0 unless $m|h$. The fact that G is strongly connected and periodic implies each node has an arc leaving it (and entering it). Hence there is an arc from node x to some node y. If $m|h$ the {y,x} entry of M^h must be 0. Suppose otherwise. Then there is a path of length m·k from y to x with k a positive integer. Now if we transverse the arc from x to y and then adjoin the path from y to x, then we have a path of length m·k + 1 from x to itself. This contradicts that fact that the period of the graph is m, hence the {y,x} entry of M^h is 0. We still have to prove that if a graph is not periodic then for some power r, the matrix M^r has no zero entry. Consider the entry {i,i}. There must be paths from i to i of two lengths, r and s, that are relatively prime. Otherwise all paths from i to i are multiples of some constant greater than 2, and then the node is periodic and so is the graph (see comment above). By Bezout's lemma we can concatenate the paths of length r and s to get paths of arbitrary length k = ar + bs. However, this does not count if either a or b is negative. It is not difficult to show that for sufficiently large integers that a positive a and b can be found. Similarly for a sufficiently large integer N there is a path from any vertex back to itself for any length greater than N. Since the graph is strongly connected there is a path from any vertex i to any other vertex j of length less than n. Hence **all** powers of the matrix of length N + n or greater will have no zero entries. Note that we can prove from elementary properties of matrices that once a power u exists such that M^u has no zeros then

M^k contains no 0's for $k \geq u$.

1.

$$A := \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 4 & 1 \\ 6 & -2 \\ 1 & 0 \end{pmatrix} \qquad C := \begin{pmatrix} 3 & -2 & -6 & 1 \\ -2 & 4 & -1 & 5 \end{pmatrix} \qquad D := \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A \cdot B = \begin{pmatrix} -9 & 8 \end{pmatrix} \qquad B \cdot C = \begin{pmatrix} 10 & -4 & -25 & 9 \\ 22 & -20 & -34 & -4 \\ 3 & -2 & -6 & 1 \end{pmatrix} \qquad C \cdot D = \begin{pmatrix} 2 & 5 & 2 \\ 9 & 8 & -1 \end{pmatrix}$$

$A \cdot C =$

array size mismatch

$$D \cdot B = \begin{bmatrix} 5 & -2 \\ 6 & 2 \\ -4 & -1 \\ 13 & -4 \end{bmatrix}$$

A·C is undefined because the array sizes are mismatched.

2.

$$M = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \qquad N = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

3.

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

4.

$$M := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad M^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad M^3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M^5 = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix} \qquad M^6 = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \end{bmatrix}$$

5. If the inverse is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and that gives us four equations: a = 1; 0 = 0; c = 1; 0 = 1. That fourth equation is a little too tough to solve! The inverse does not exist.

6.

$$A := \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C := \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 1 & -0.5 \\ 0 & 1 & -2 & 1.5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$