26

Applications of Probability

The Binomial Distribution

The binomial distribution is as important as any distribution in probability. It is quite simply the description of the outcome of throwing a coin n times. The binomial coefficient graph of Section 15 is reproduced here as **Figure 1** with only slight modification. Each node is an intersection. We start at the top node which is on level 0 and proceed to higher levels by making left and right turns at each level. Any intersection on a particular level can be characterized by the number of left turns it takes to get there. For example, the second intersection from the left on level 3 is denoted by the binomial coefficient $\binom{3}{2}$ because it is on the third level and it takes 2 left turns to get there (out of 3 turns). In Chapter 5 we equated this coefficient with the number 3, because there are exactly 3 ways to get to that intersection. That is, there are 3 ways to go from the node on level 0 to the node $\binom{3}{2}$ on level 3, depending on whether you go left-left-right, left-right-left, or right-left-left.

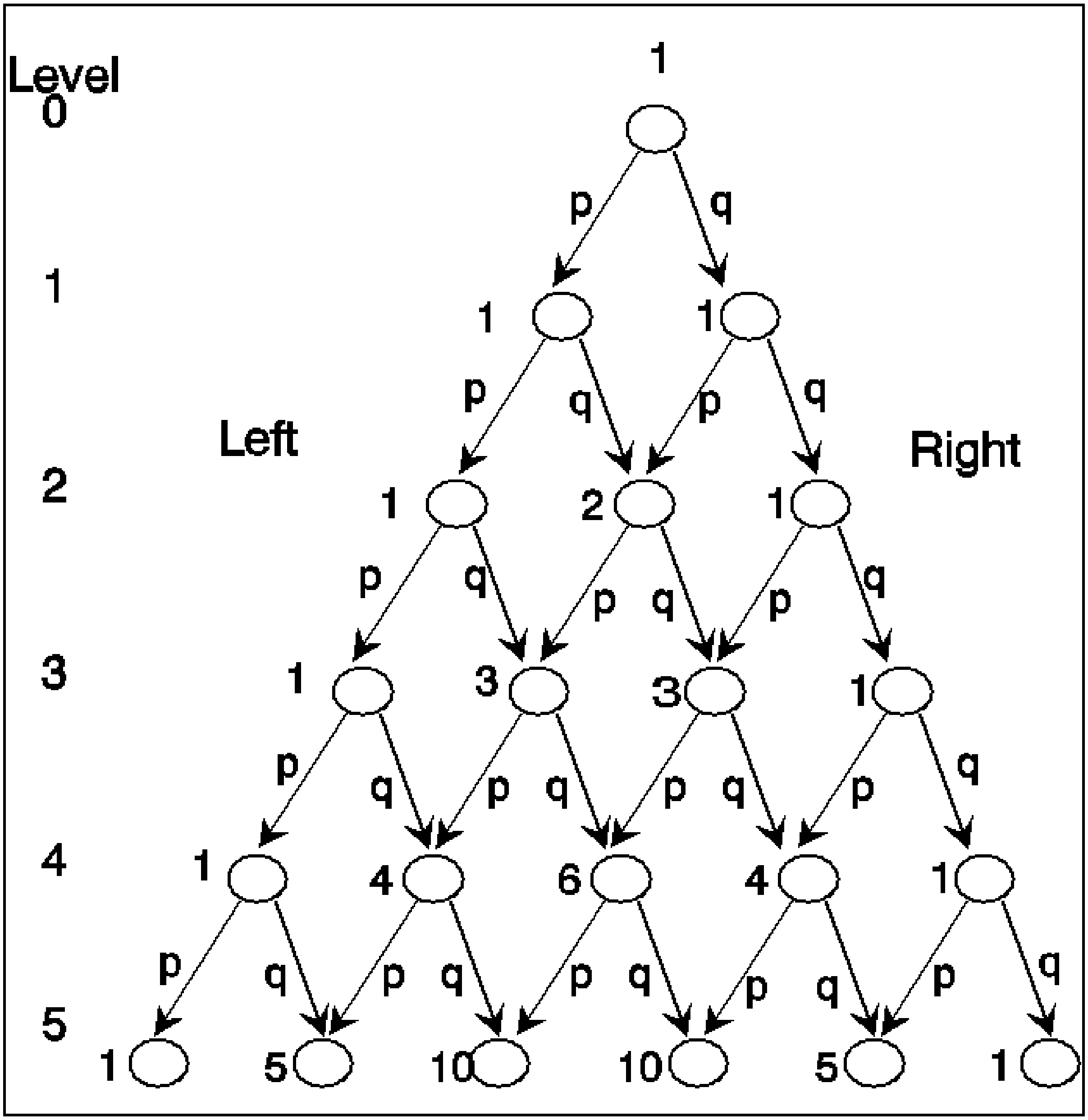

Figure 1 The Binomial Graph With Probabilities

Now suppose we change the problem. At any given node we go left with probability p, and right with probability q, where p + q = 1. The question is now, not in how many ways can we get to node $\binom{3}{2}$ (also denoted earlier as [3,2]) but what is the probability of reaching there (after starting at the intersection at the top)? Let us consider the route, left-left-right. It has probability of $p \cdot p \cdot q = p^2q$ of occurring. Similarly, the route left-right-left has probability $p \cdot q \cdot p = p^2q$ of occurring. The third route, right-left-left, has probability $q \cdot p \cdot p = p^2q$ of occurring. That is, each route has the same probability of occurring, and there are three possible routes, so the total probability of reaching node $\binom{3}{2}$ after starting at the top is $\binom{3}{2}p^2q = 3p^2q$. In general, the probability of reaching node $\binom{n}{m}$ is $\binom{n}{m}p^mq^{n-m}$.

> Throughout this chapter, we make use of the properties of independence. Suppose that I make a fair toss of a coin whose probability of heads is .8 and of tails is .2. (Obviously, the coin is unfair although the toss isn't.) If I do two throws, the probability that I throw heads and then throw tails is $.8 \cdot .2 = .16$. I can multiply the probabilities to get their joint probability because the throws are independent. This is the third property of independence given earlier.

Any left-right, on-off, yes-no, success-failure type of experiment is known as a *Bernoulli experiment* or *Bernoulli trial*.[1] A sequence of identical and independent Bernoulli trials is a *binomial experiment*. The first such example in virtually every textbook is flipping a coin four times.[2] Let us suppose that my coin is biased with probability of heads being .6 (completely to my own surprise). We might ask the question, if I flip the coin four times, what is the probability of exactly 2 heads? To qualify as a binomial experiment each throw must be independent of each other throw. I can achieve this by doing *fair* throws (although the coin itself

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[1]The Bernoullis were an eighteenth century family of mathematicians spanning three generations. They are the greatest family of mathematicians ever (and were frequently at one another's throats). One test of the true mathematical historian is knowing which Bernoulli is which. (I flunk.)

[2]Notice that I saved it until the second example!

is not fair). In binomial experiments, we traditionally speak arbitrarily of one result as a success and the other as a failure. Let us call heads a success. In the language of probability and statistics, we are asking for the probability of exactly two successes in 4 Bernoulli experiments where probability of success is .6. There are $\binom{4}{2} = 6$ ways to pick 2 out of 4 tosses as heads.

These are:

- ▹ HHT
- ▹ THT
- ▹ HTH
- ▹ TTH
- ▹ THT
- ▹ HTH
- ▹ HTT
- ▹ THH

Each of these combinations has probability $.6^2.4^2$, so that the total probability of throwing exactly 2 heads is $6\cdot.6^2.4^2 = .3456$.

In n independent Bernoulli trials, the probability of exactly m successes, where p is the probability of success and q = 1 − p is the probability of failure, is:

$$\binom{n}{m} p^m q^{n-m}$$

Formula 1 The Binomial Probability Formula

Note that if we use the binomial expansion formula (Chapter 5) to evaluate $(p+q)^n$ each term is of the form $\binom{n}{m} p^m q^{n-m}$. The sum of the terms is 1 since p + q = 1.

Example If Repunzel does 100 fair flips of a fair coin, we know by considerations of symmetry that the most likely outcome is 50 heads. However, exactly how likely is that? By the binomial probability formula the answer is $\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50}$ and, using a calculator this turns out to be roughly .079589. Note that when p = .5 the binomial formula simplifies to $\binom{n}{m}\left(\frac{1}{2}\right)^{n}$.

☐ **Exercise 1** Prove the assertion that was just given.

Example Suppose that we flip a coin, whose probability of heads is .9, 10 times. We could ask what is the probability of 2 or more heads. To do this, we could, use the binomial formula 8 times to calculate the exact probability of 2 heads occurring, 3 heads occurring, though 10 heads occurring, and then add up the answers. A simpler way of answering the problem is to realize that the event that 2 or more heads occur, is the complement of the event that 0 or 1 heads occur. (Remember: the complement of X is *not X*.) The probability of 0 or 1 heads occurring is: $\binom{10}{0}(.9)^0(.1)^{10} + \binom{10}{1}(.9)^1(.1)^9$ and this is $1\cdot.1^{10} + 10\cdot.9\cdot.1^9 = .0000000091$. The probability of the complement of 0 or 1 heads occurring is 1 − .0000000091 = .9999999909.

For the following exercises assume that we have tossed a coin 7 times, where the probability of tossing heads is .6. Use the binomial probability formula for all exercises.

☐ **Exercise 2** What is the probability of tossing 7 heads?

☐ **Exercise 3** What is the probability of tossing 0 heads?

☐ **Exercise 4** What is the probability of tossing 4 heads?

☐ **Exercise 5** What is the probability of tossing 5 heads?

☐ **Exercise 6** What is the probability of tossing 4 or 5 heads?

☐ **Exercise 7** What is the probability of tossing 0 or 1 heads?

☐ **Exercise 8** What is the probability of tossing 2 or more heads?

☐ **Exercise 9** What is the probability of tossing 7 heads given that you toss 2 or more heads?

☐ **Exercise 10** What is the probability of tossing 4 or 5 heads given that you toss 2 or more heads?

The Recursive Binomial Probability Formula

Remember, that earlier we had a recursive formula for binomial coefficients. It was: $\binom{n}{r} = C(n,r) = \frac{n}{r}C(n-1,r-1)$ with C(n,0) = C(n,n) = 1. It is reasonable to expect that this formula can be extended for binomial probabilities. This is particularly important if calculating a table of binomial probabilities; it enables one to calculate each subsequent probability with a minimum of work.[1] Let us denote the probability of r successes in n Bernoulli trials, each with probability of success, p, by B(r; n, p). Then:

[1]The **best** way to calculate a binomial probability table is by using a spreadsheet. The formula conventions in spreadsheets are ideal for recursion.

$$\mathbf{B}(0;\ \boldsymbol{n},\ \boldsymbol{p}) = (1-\boldsymbol{p})^{\boldsymbol{n}}$$

$$\mathbf{B}(\boldsymbol{r};\ \boldsymbol{n},\ \boldsymbol{p}) = \frac{\boldsymbol{n}}{\boldsymbol{r}}\boldsymbol{p}\,\mathbf{B}(\boldsymbol{r}-1;\ \boldsymbol{n}-1,\ \boldsymbol{p}); \qquad \boldsymbol{r}>0$$

Formula 2 The Recursive Formula for Binomial Probabilities

The Geometric Distribution

A probability distribution closely related to the binomial distribution is the geometric distribution. In both cases you are performing repeated Bernoulli trials. That is, you are doing repeated independent identical experiments with two outcomes. Typically, you arbitrarily call one outcome success and the other failure. In the binomial case, you do n experiments. The binomial probability formula tells you the probability of exactly m successes in those n experiments. In the geometric case, you do not know how many experiments you are going to do. You simply keep doing experiments until you have one success. The geometric probability distribution gives the probability that success occurs on the nth trial.

As before let us denote the probability of a success by p. Then the probability of failure is $q = 1 - p$. Since the experiments are independent, the probability that the first r experiments will be failures is q^r. The probability that the first success will occur on experiment n is $q^{n-1}p$. This might be clearer from a concrete example. Suppose, that our experiment is to toss a die until we get a 6. The question that we want to answer is: what is the probability that the first 6 occurs on the nth throw. Clearly the probability that the first throw is a success is 1/6. The probability that the first success occurs on the second throw is the probability that the first throw is a failure (5/6) times the probability that the second throw is a success (1/6). The probability that the first success occurs on the third throw is the probability that the first and second throws are failures $(5/6)\cdot(5/6) = (5/6)^2$ times the probability that the third probability is a success (1/6) giving an overall probability of $(5/6)^2(1/6)$. We can see that the probability that the first success occurs on trial n, is $\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$, as given by the formula above. Note that by this line of reasoning we can conclude that the probability that first success occurs **after** throw n is $\left(\frac{5}{6}\right)^n$.

That is: **The case that the first success occurs after throw n, is precisely the case that the first n throws are failures.**

> **If performing Bernoulli experiments, with probability of success p, the probability that the first success occurs on trial n is:**
>
> $$q^{n-1}p \;=\; (1-p)^{n-1}p$$

Formula 3 The Geometric Distribution

For the following exercises, assume that you are doing a fair flip of a fair coin:

☐ **Exercise 11** What is the probability that the first heads occurs on throw 3?

☐ **Exercise 12** What is the probability that the first heads occurs on throw 4?

☐ **Exercise 13** What is the probability that the first heads occurs on throw n?

☐ **Exercise 14** What is the probability that the first heads occurs after throw n?

Expected Values

Expected values are both a simple and meaty subject. (Pay close attention.) A recurring problem in math education, is that students underestimate those concepts that seem easy (and often are easy). This is one of those concepts.

Expected value is a concept closely related to arithmetic average. You can think of the expected value of a situation to be its average outcome. Suppose we have a collection of n exclusive and exhaustive outcomes, x_1, x_2, x_3, ... , x_n. Suppose that the probability of each outcome x_i is p_i. Since the outcomes are exclusive and exhaustive, we have $p_1 + p_2 + p_3 + ... + p_n = 1$.

Given n exclusive and exhaustive outcomes, x_1, x_2, x_3, ... , x_n each with probability p_i the expected value of the x_i's is given by:

$$E(x_i) = \sum x_i \cdot p_i = x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

That is, each outcome is multiplied by its probability and their sum is the expected value.

Formula 4 The Definition of Expected Value

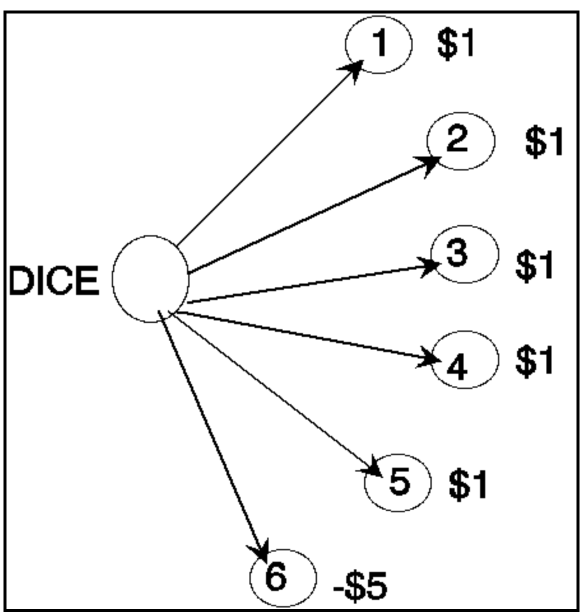

Figure 2 A "Fair" Dice Game

Consider a simple game played by Fred and Repunzel. They throw a fair die. Each time they throw a 1 through 5 Fred pays Repunzel $1, and every time they throw a 6, Repunzel pays

Fred \$5. There game is represented by the graph in **Figure 2**. The graph shows six outcomes, but it is simpler to view the game as having two outcomes, F, Fred wins, and R, Repunzel wins. We have P(F) = 1/6 and P(R) = 5/6. **From Repunzel's viewpoint**, the expected outcome, or the expected value of the outcome is:

$$\mathrm{E}(\mathit{Repunzel's\ winnings}) = \frac{5}{6}(1\$) + \frac{1}{6}(-\$5) = 0$$

The expected value of the game for Repunzel (and also for Fred) is 0, so this is what we call a *fair game*. In the long run the average winnings for Repunzel should be \$0. This is exactly what we expect if we think about it. Consider the case where Repunzel and Fred play 60 games. Repunzel should win 50 times for \$50 and he should lose 10 times for −\$50 and a total of 0\$.

If we make a fair throw of a fair die, the expected value of the outcome is: $(1/6)\cdot 1 + (1/6)\cdot 2 + (1/6)\cdot 3 + (1/6)\cdot 4 + (1/6)\cdot 5 + (1/6)\cdot 6 = 3.5$. Again, 3.5 is the average outcome if we throw a die many times. On the other hand, the expected value 3.5 cannot occur on any particular throw. Similarly, if we look at the fair flip of a fair coin, if the value of heads is 1 and of tails is −1 then the expected outcome is 0. If however, we decide to record heads as 1 and tails as 0, then the expected outcome is ½.

It is easy to say that in an investment we would like an expected value to be positive. Why make an investment with negative expected value? On the other hand, the expected value should only be viewed as one measure, and its interpretation can be oversimplified. For example, when we purchase insurance, the expected value is negative or otherwise the insurance company could not make a profit. We accept the negative expected value in order to reduce our anxiety and the probability of suffering a catastrophic loss..

For the following exercises assume that we are doing a fair throw of two die.

☐ **Exercise 15** What is the expected sum of the die?

☐ **Exercise 16** Suppose one die is red and the other is green. Let us call the value of the red die minus the value of the green die *the difference*. What is the expected value of the difference? What is the expected value of the absolute value of the difference? (If the difference is -2 then the absolute value of that difference is 2.)

☐ **Exercise 17** What is the expected value of the maximum of the two die? What is the expected value of the minimum of the two die? For example if one die is 3 and the other is 5, their minimum is 3 and their maximum is 5.

☐ **Exercise 18** What is the expected value of the product of the two die?

Example Usually when people take a multiple choice test, they guess answers they do not have time to read. If the test has four answers per problem the test taker can expect to gain ¼ a point for each guessed answer. Therefore some tests penalize the test-takers for incorrect answers. For each incorrect answer, x points are taken off, where x is chosen so that the expected value for guessed problems is 0. Therefore x must satisfy: $\frac{1}{4}\cdot 1 - \frac{3}{4}\cdot x = 0$. Solving for x, we get x = ⅓ points.

Example[1] American roulette is a sucker game. When you bet on a number, your chance of hitting the number is 1/38 but you pay-off is only \$35. Hence the expected loss on each bet is $(1/38)\cdot 35 + (37/38)\cdot(-1) = -.0526$. That is you lose 5.26 cents on each dollar bet.[2] Mutt plays the tables everyday. He always bets on number 00

[1] This problem appears as problem 7 in Fifty Challenging Problems in Probability with Solutions., by Frederick Mosteller. This is a Dover Paperback published in 1987. It is available for around \$5. I have changed the particulars.

[2] It turns out that the other types of bets in American roulette all have the same expected values.

exactly 36 times. His buddy Jeff wants to cure Mutt of this habit, so he makes a side wager with Mutt. He bets Mutt \$20 that at the end of the 36 bets he is behind. Our question is, what is Mutt's expected return on the \$36 he bets each day when we factor in the side bet? His expected return against the roulette table is 36 times his return for each bet; that is $36\cdot(-.0526) = -1.8947$. His average outcome per day against the tables is -1.89\$. But if he loses on the day he owes Jeff \$20 otherwise Jeff owes him. Note that if Mutt wins exactly one bet against the table, he comes out ahead since he keeps the dollar he bet and he makes \$35. He loses the other \$35 and he is up by exactly one dollar. To come out behind he must lose all 36 bets, and the probability of that is $(37/38)^{36} = .3829$. So there is only a .38 probability that Mutt pay Jeff 20\$ and there is a .62 probability that Jeff pays Mutt \$20. The total expected return for Mutt is $-1.89 + .38(-20) + .62(20) = 2.79$. Hence, before the side-bet, Mutt could expect to lose 1.89\$ per day. But with the side-bet, he can expect to average \$2.79 profit per day.

An Example of a Great Betting Opportunity[1]

Let us suppose that we have an uncommonly good opportunity. We have \$100 and we can bet as often as we like as follows:

Each bet is independent of the other bets and has a 50% of winning and losing. If a bet is won, it returns \$1.40, *profit*, for each dollar bet. In other words if you have \$1 and you bet it and win, then you have \$2.40. If you lose, you lose the money bet. When you bet, you must bet one-half of your capital. For example, your first bet would be \$50.

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[1]This example is based upon the first example in *The Art of Decision Making* by Morton Davis (Springer-Verlag, 1986). Davis' book is an excellent introduction to decision analysis. It is a short sweet sampling of problems and is easily suited to the level of anyone who has finished this book. As for this particular problem, I have made a few small changes. I wanted to make it different.

Let us ask, from an expected values point of view, how many bets should we make? If we have \$X, we bet ½X and we end up with either ½X or $X + ½X·1.40 = X(1.7)$. The expected outcome of a bet is (in terms of our capital X) $E(X) = ½·½X + ½·(1.7)X = 1.1X$. In English: our expected capital is a 50% probability of reducing our capital by one-half and a 50% of increasing our capital by 70%. The expected outcome is that we have increased our capital by 10%. Notice, that this is independent of the size, X, of our capital. Hence, if we think only in terms of expected values, we should bet as often as we can. If we bet n times, starting with \$X, our expected capital would be $(1.1)^n·X$. For example, starting with \$100, if we make 200 bets, the expected size of our stash after making the bets is $(1.1)^{200}·100 = \$18,990,527,646$.

It seems, from an expected values point-of-view, that the bet is a fantastic opportunity. Be assured that the preceding analysis is correct as far as it goes. That is, suppose we start with \$100 and do 200 bets, and suppose further we do this 500,000 times, starting with \$100 each time (we have time to spare). Then the arithmetic average of the outcomes would be very close to \$18,990,527,646. There has been no trick; this is correct.

Let us look at the case where we make 100 bets and we win 50, which of course is the most likely outcome. We start with \$100 and we win fifty bets, each of which in effect multiplies our capital by 1.7 and we lose fifty bets, each of which in effect multiplies our capital by ½. Notice, that if we multiply a number, X, by the numbers 1.7 and ½, it doesn't matter in which order we multiply.[1] Therefore, in this case, we start with \$100 and we wind up with $100·(1.7)^{50}(.5)^{50} = 100·(1.7·.5)^{50} = 100·(.85)^{50}$. That is a win and a loss is the same as multiplying by 1.7 and .5 for a net effect of multiplying by .85. If this happens fifty times we are multiplying by $(.85)^{50} = .00029576$. After starting with \$100 we wind up with ¢.03 (not 3 cents, but .03 cents!).

To recapitulate, we have a betting opportunity with an enormous expected value or average outcome, but in the most likely case, we wind up totally broke. (What the &%^& is

[1]This is just the law of commutivity of multiplication which you saw in high school algebra.

going on?) Precisely what is the probability that we come out ahead if we make 100 bets?[1] Since each win multiplies our capital by 1.7 and each loss multiplies our capital by .5; to break even, we need to end up multiplying by 1. That is, we need to solve: $(1.7)^r(.5)^{100-r} = 1$ where r is the number of bets out of 100 we must win to break even. To solve this for r, we take the logarithm of both sides (any base) to get $\log(1.7)^r + \log(.5)^{100-r} = 0$ (since $\log(1) = 0$). $r\cdot\log(1.7) + (100-r)\cdot\log(.5) = 0$ (since $\log(x^y) = y\cdot\log(x)$). $r\cdot\log(1.7) - (100-r)\cdot\log(2)$ (since $\log(½) = -\log(2)$). $r\cdot(\log(1.7) + \log(2)) = 100\cdot\log(2)$ (by rearrangement). Solving for r, we get $r = 56.6$. That is, out of 100 bets we need to win more than 56 just to come out ahead[2] and the chance of that is just under 10%.

The explanation for the above is quite simple. When we come out ahead we come out so far ahead that it makes our average outcome huge. If we were only to make 2 bets (starting with $100) we have the following equally likely outcomes:

- Win-Win: $100\cdot 1.7\cdot 1.7 = 289$
- Win-Loss: $100\cdot 1.7\cdot .5 = 85$
- Loss-Win: $100\cdot .5\cdot 1.7 = 85$
- Loss-Loss: $100\cdot .5\cdot .5 = 25$

We come out ahead only once in four times but our "average" outcome is $¼(289) + ¼(85) + ¼(85) + ¼(25) = ¼(289 + 85 + 85 + 25) = 121$: a profit of $21.

An important Modification of the Great Betting Opportunity

Suppose that we take the preceding bet and allow a modification. We will allow you to bet whatever proportion of your capital you like.

[1]This part is a little more analytical than most of the book. If logarithms scare you, just skim this section and go on. (But you really should not be scared off by logarithms).

[2]The standard way to figure the probability of this (instead of summing binomial probabilities) is to approximate the binomial distribution by the normal distribution. Although that is simple to do, it is outside the scope of this text. (I summed the binomial however.)

Analysis will show that the higher the proportion you bet, the higher the expected return. Also, the higher the proportion you bet, the lower the probability that you come out ahead.

In the extreme case, you bet 100% of your capital. In 100 bets, the only way that you can come out ahead is if you win every bet. The probability of that is $(.5)^{100} = 7.8886\times10^{-31}$. However, in that one case we make $\$100\cdot(2.4)^{100}$. The expected value is then $(.5)^{100}\cdot100\cdot(2.4)^{100} = 100\cdot(1.2)^{100} = 8{,}281{,}797{,}452.20$. At this extreme, the probability of coming out ahead is absolutely insignificant, but on the average we make a huge fortune.

We can ask, what is the proportion of resources we must bet to have a 50% chance of coming out ahead? Using similar techniques to those above, the answer is 28.57%.[1]

Suppose that we have the above betting opportunity: 50% chance of winning each bet, and \$1.40 profit on each bet won. Suppose further that this time you can bet whatever proportion of your capital you choose. What should you do? This is one example, out of many, that come up in probability, where there is no correct answer. Each person can have their own philosophy and there is no general mathematical solution to who is correct. If it were my betting opportunity, and I were starting with only \$100, I would start conservatively betting no more than 20% of my capital. If and when my capital grew, I might let the proportion edge up to 27%.

[1]Suppose that the proportion of our capital that we are betting is p, $0 \le p \le 1$. We want to find p such that if we win half of our bets we are even. That is, if we make 2r bets, we have: $(1+1.4p)^r\cdot(1-p)^r = 1$. $r\cdot\log(1+1.4p) + r\cdot\log(1-p) = 0$. $\log(1+1.4p) + \log(1-p) = 0$. $\log(1 + .4p - 1.4p^2) = 1$. $1 + .4p - 1.4p^2 = 1$. $p(.4 - 1.4p) = 0$. $p = 0$; $p = .4/1.4 = .2857$. Notice that $p = 0$ is also a valid, though trivial, solution.

A Technique for Doing Statistics Problems

(This section is optional)

This book is intended to introduce to probability (amongst other things) and to prepare you for statistics, not to teach any statistics. However, the line between probability and statistics can be very fine.

The binomial distribution is absolutely essential to statistics. Many tests come down to a succession of success-failure type experiments. Let us consider a typical problem. Repunzel does an ESP test of JeNifer. Repunzel asks JeNifer the suits of all 52 cards in a well-shuffled deck of ordinary playing cards. They go through the entire deck and at each card Repunzel records the result but gives no indication to JeNifer of her score.[1] Since there are four equally likely suits, the expected result would be for JeNifer to score 13 out of 52 correct. However, JeNifer scores 17 out of 52. Our question is, did JeNifer do so well just by luck, or could there be other explanations.[2]

The standard approach to the above question is to ask ourselves, how likely is it that JeNifer could have done at least that well just by luck. That is, *assume* that she only has luck. What is the probability that she should guess 17 or more correct? We do **not** ask what is the probability that she should guess exactly 17 correct just by luck, because that is too restrictive. Guessing exactly 13 correct has a fairly small probability (.127) even though that is the most likely case. Formally, we assume the *Null Hypothesis* that JeNifer only has luck and we test how likely the outcome was under that assumption. Again, we are testing the outcome of 17 *or more* correct. In other types of problems we might test the probability of missing 13 correct by 4 or more; that is what is the probability of 17 or more correct or 9 or fewer correct. What test we

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[1]In practice this is quite difficult and requires a fairly strict protocol (procedure), however, Repunzel and JeNifer are amateurs and ignore all of that.

[2]The other explanation that Repunzel and JeNifer are looking for is ESP. However, a much more likely explanation is that they simply did not conduct the experiment carefully enough; Repunzel inadvertently gave JeNifer nonverbal clues (see previous footnote).

conduct depends on the problem and can be quite judgmental. In the case, we are interested in the probability of doing *better* than expected. Usually we do not worry that someone did poorer than expected because of ESP.[1]

It turns out that the probability that someone would guess 17 or more correctly just by luck is roughly .1322. We could reject that null hypothesis with a significance level of .1322, or we could say *with confidence of 85%* (since we did not quite reach 90%). Unfortunately for Repunzel and JeNifer, nobody is impressed by that. 90% confidence is considered minimal and in this sort of thing people want 99% confidence.

The reason, that statistics seems complicated is that it takes a while to digest all of these concepts, and it is not trivial to estimate the probability of achieving a certain score by luck. There are binomial tables for answering such questions, but even reading them properly can give some people difficulty, and they will often not cover the case you are interested in. The more usual procedure is to use the approximation to the binomial by the Normal distribution, which requires translating the problem to the Normal distribution and then using a Normal table.

However, we can avoid all of this table reading. The above methods were created before computers were in use. We can estimate the probability of guessing 17 or more cards correctly by simulating the experiment by computer. We write a program that simulates making 52 guesses each with a .25 probability of success. We have the program do this say 1000 times, and we record the proportion of cases where the program guessed correctly 17 or more times. If you know how to program, this program can be written in a language such as Pascal or even Basic in roughly 15 minutes. On a modern microcomputer you can get an answer within 5 minutes of starting the program. This sort of method is known as *computationally intensive*. Until recently it was considered impractical. However, computationally intensive techniques have three great advantages over traditional techniques.

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[1]Although in fact, ESP researchers do worry about just that. Most ESP researchers do not ever seriously consider the possibility that ESP does not exist or that someone does not have it. ESP research has provided many instructive examples over the years of how not to do statistics tests.

1: Computationally intensive techniques are easier to understand conceptually and therefore facilitate learning statistics

2: Computationally intensive techniques are easy to implement on today's computers.

3: Computationally intensive techniques often require fewer assumptions than traditional techniques. Therefore the results are often more meaningful and convincing than results from traditional methods.

1. With p = ½ and q = ½ the binomial distribution formula is:

$$\binom{n}{m}.5^m.5^{n-m} = \binom{n}{m}.5^n$$

   The last step follows from the law of exponents.

2. .02799

3. .00164

4. .29030

5. .26127

6. This is just the sum of the last two answers: .55158

7. P(0) + P(1) = .00164 + .01720 = .01884

8. P(≥2) = 1 − (P(0) + P(1)) = .98116

9. P(7 | ≥2) = P(7 and ≥2)/P(≥2) = P(7)/P(≥2) = .02853

10. P(4 or 5 | ≥2) = P((4 or 5) and ≥2)/P(≥2) = P(4 or 5)/P(≥2) = .56217

11. ⅛

12. 1/16

13. $(.5)^{n-1}\cdot(.5) = (.5)^n$

14. $(.5)^n$. This can be proven in two ways. First if we set up the infinite sum, it turns out to be a geometric series for which the formulas were given earlier. The second solution is to observe that the only way that the first heads can occur after throw n is precisely the case that the first n throws are tails.

15. (1/36)2 + (2/36)3 + (3/36)4 + (4/36)5 + (5/36)6 + (6/36)7 +
    (5/36)8 + (4/36)9 + (3/36)10 + (2/36)11 + (1/36)12 = 7

16. This is sort of a trick question. Notice that for every difference a−b there is an equally likely b−a. Hence the expected difference is 0. The expected absolute difference is 35/18. This is solved by brute force: that is by calculating all of the cases.

17. The expected value of the max is 161/36. The expected of the min is 91/36. Note that the expected value of the min plus the absolute difference gives you the max!!!!

18. 12.25. This can be solved by simply working out all of the cases. However, a theorem, not used in this book, states that the expected value of the product of two independent variables is the product of their expected values.