The EOQ Inventory Formula

James M. Cargal
Mathematics Department
Troy University – Montgomery Campus

A basic problem for businesses and manufacturers is, when ordering supplies, to determine what quantity of a given item to order. A great deal of literature has dealt with this problem (unfortunately many of the best books on the subject are out of print). Many formulas and algorithms have been created. Of these the simplest formula is the most used: The EOQ (economic order quantity) or Lot Size formula. The EOQ formula has been independently discovered many times in the last eighty years. We will see that the EOQ formula is simplistic and uses several unrealistic assumptions. This raises the question, which we will address: given that it is so unrealistic, why does the formula work so well? Indeed, despite the many more sophisticated formulas and algorithms available, even large corporations use the EOQ formula. In general, large corporations that use the EOQ formula do not want the public or competitors to know they use something so unsophisticated. Hence you might wonder how I can state that large corporations do use the EOQ formula. Let's just say that I have good sources of information that I feel can be relied upon.

The Variables of the EOQ Problem

Let us assume that we are interested in optimal inventory policies for widgets. The EOQ formula uses four variables. They are:

- D: The demand for widgets in quantity per unit time. Demand can be thought of as a rate.
- Q: The order quantity. This is the variable we want to optimize. All the other variables are fixed quantities.
- C: The order cost. This is the flat fee charged for making any order and is independent of Q.
- h: Holding costs per widget per unit time. If we store x widgets for one unit of time, it costs us $x \cdot h$.

The EOQ problem can be summarized as determining the order quantity Q, that balances the order cost C and the holding costs h to minimize total costs. The greater Q is, the less we will spend on orders, since we order less often. On the other hand, the greater Q is the more we spend on inventory. Note that the price of widgets is a variables that does not interest us. This is because we plan to meet the demand for widgets. Hence the value of Q has nothing to do with this quantity. If we put the price of widgets into our problem formulation, when we finally have finally solved the optimal value for Q, it will not involve this term.

The Assumptions of the EOQ Model

The underlying assumptions of the EOQ problem can be represented by **Figure 1**. The idea is that orders for widgets arrive instantly and all at once. Secondly, the demand for widgets is perfectly steady. Note that it is relatively easy to modify these assumptions; Hadley and Whitin [1963] cover many such cases. Despite the fact that many more elaborate models have been constructed for inventory problem the EOQ model is by far the most used.

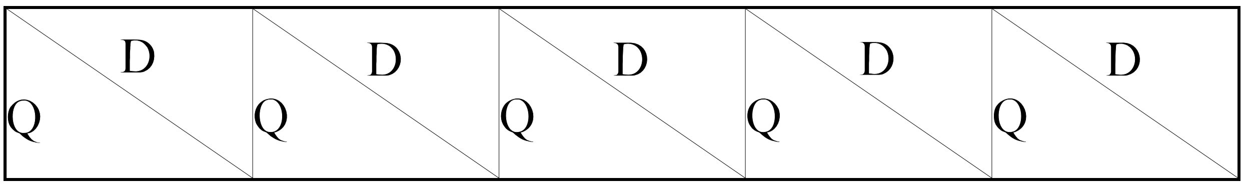

Figure 1 The EOQ Process

An Incorrect Solution

Solving for the EOQ, that is the quantity that minimizes total costs, requires that we formulate what the costs are. The *order period* is the block of time between two consecutive orders. The length of the order period, which we will denote by P, is Q/D. For example, if the order quantity is 20 widgets and the rate of demand is five widgets per day, then the order period is 20/5, or four days. Let T_p be the total costs per order period. By definition, the order cost per order period will be C. During the order period the inventory will go steadily from Q, the order amount, to zero. Hence the average inventory is Q/2 and the inventory costs per period is the average cost, Q/2, times the length of the period, Q/D. Hence the total cost per period is:

$$\mathrm{T}_P = C + \frac{Q}{2} h \frac{Q}{D} = C + \frac{Q^2 h}{2D}$$

If we take the derivative of T_p with respect to Q and set it to zero, we get Q = 0. The problem is solved by the device of not ordering anything. This indeed minimizes the inventory costs but at the small inconvenience of not meeting demand and therefore going out of business. This is what many people, perhaps most people do, when trying to solve for the EOQ the first time.

The Classic EOQ Derivation

The first step to solving the EOQ problem is to correctly state the inventory costs formula. This can be done by taking the cost per period T_p and dividing by the length of the period, Q/D, to get the total cost per unit time, T_u:

$$\mathrm{T}_\mathrm{u} = \frac{CD}{Q} + \frac{Qh}{2}$$

In this formula the order cost per unit time is CD/Q and Qh/2 is the average inventory cost per unit time. If we take the derivative of T_u with respect to Q and set that equal to 0, we can solve for the economic order quantity (where the exponent * implies that this is the optimal order quantity):

$$Q^* = \sqrt{\frac{2CD}{h}}$$

If we plug Q^* into the formula for T_u, we get the optimal cost, per unit time:

$$T_u^* = \sqrt{2CDh}$$

The Algebraic Solution

It never hurts to solve a problem in two different ways. Usually each solution technique will yield its own insights. In any case, getting the same answer by two different methods, is a great way to verify the result. If we multiply the formula for T_u on both sides by Q, we get, after a little rearranging, a quadratic equation in Q:

$$\frac{Q^2 h}{2} - QT_u + CD = 0$$

Next we divide through by h/2 to change our leading coefficient to 1. Completing the square, we get:

$$\left(Q - \frac{T_u}{h}\right)^2 + \frac{2CD}{h} - \frac{T_u^2}{h^2} = 0$$

Note that this equation has two variables. T_u is a function of Q (and could well be written as $T_u(Q)$). Hence we have a curve in the Cartesian plane with axes labeled Q and T_u. We want the value of Q that minimizes T_u. Notice that the term $\frac{2CD}{h}$ is a constant. Writing that constant as k, and rearranging the equation, we get:

$$\frac{T_u^2}{h^2} = \left(Q - \frac{T_u}{h}\right)^2 + k$$

If we set the quadratic term to zero, then $T_u = h\sqrt{k}$. Any change in the quadratic term from zero increases the size of T_u. Hence the optimal size of T_u is $h\sqrt{k}$ which just happens to be $\sqrt{2CDh}$ which is the value we found earlier. The quadratic term is zero if and only if $Q = \frac{T_u}{h}$. This gives us the identity $Q^* h = T_u^*$.

An Example

It is useful at this point to consider a numerical example. The demand for klabitz's is 50 per week. The order cost is $30 (regardless of the size of the order), and the holding cost is $6 per klabitz per week. Plugging these figures into the EOQ formula we get:

$$Q^* = \sqrt{\frac{2 \cdot 30 \cdot 50}{6}} = 22.36$$

This brings up a little mentioned drawback of the EOQ formula. The EOQ formula is not an integer formula. It would be more appropriate if we ordered klabitz's by the gallon. Most of the time, the nearest integer will be the optimal integer amount. In this case, the total inventory cost T_u is $134.18 per week when we order 22 klabitz's. If instead, we order 23 klabitz's the cost is $134.22.

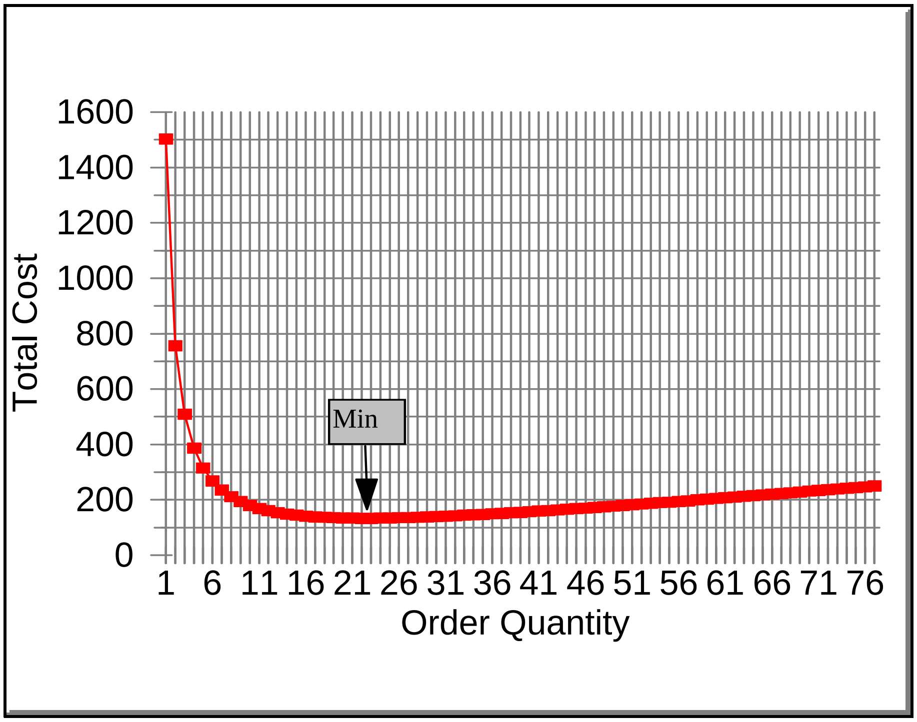

Figure 2 Total Cost as a Function of Order Quantity

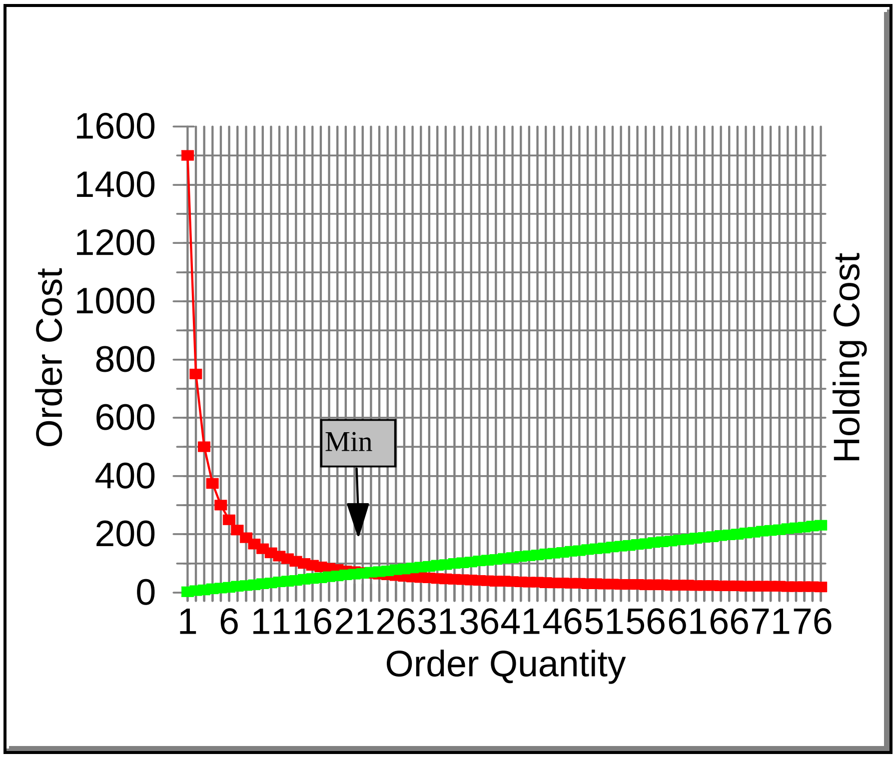

Figure 3 Order and Holding Costs

A graph of this problem is illuminating: **Figure 2**. Because the graph is so flat at the optimal point, there is very little penalty if we order a slightly sub-optimal quantity. We can better understand the graph if we view the combined graphs of the order costs and the holding costs given in **Figure 3**. The basic shapes of all three graphs (total costs, order costs, holding costs) are always the same. The graph of order costs is a hyperbola; the graph of holding costs is linear; and as a result the graph of the total costs (T_u) is convex. This can also be seen in that the function $\frac{dT_u}{dQ}$ is increasing and the function

$$\frac{d^2T}{dQ^2} = \frac{2CD}{Q^3}$$

is positive every where. If we plug the optimal quantity, Q^*, into this last formula we get:

$$\frac{d^2T^*}{dQ^2} = \frac{\sqrt{2hCD}}{4h^2CD}$$

Ordinarily this last quantity is very small, which indicates that the total cost of inventory T_u changes very slowly with Q (in the optimal region). Hence the assumptions of the EOQ model do not have to be accurate because the problem usually is tolerant of errors.

If you study closely the graphs in **Figure 3**, it may seem clear to you that their sum, T_u, reaches a minimum precisely where the two graphs intersect; that is at the point where order costs and holding costs are equal. The gives us the equality $\frac{Qh}{2} = \frac{CD}{Q}$. Solving that equality is the easiest way to derive the EOQ formula.

Why Use the EOQ Formula At All?

A problem that occurs in applied mathematics more than pure is that we hang onto formulas and techniques that have been made (mostly) obsolete by technology. Try to think what it was like to solve a problem like the one here forty years ago. The simple operation of division was either done by hand, or by use of logarithms out of a table, or less exactly by using a slide rule. It was not practical to simply calculate the total cost of inventory for a large set of order quantities and to compare answers. The EOQ formula simplified the problem to a minimal number of calculations. However, now it is quite simple to calculate total costs of inventory for hundreds of order quantities, and this can be done from scratch in less time than it use to take to employ the EOQ formula. We can do it with a spreadsheet.

The spreadsheet in **Figure 4**, takes the problem from the previous example, and computes for a large variety of quantities the order costs, holding costs, and total costs. It took about 15 minutes to set this spreadsheet up, and most of that time was spent on formatting (for example putting in lines). The formulas for order costs and holding costs were inserted to calculate the respective entries for the order quantity of one. The total cost entry is defined to be the sum of the first two entries (although its column comes first). Then the formulas are copied down the length of the order quantity column. The only trick is that you must remember to use absolute addresses for the fixed parameters. Once the spreadsheet is set up, the values for C, D, and h can be changed and the entire spreadsheet will recalculate in less than a second. Note also, that the EOQ formula itself is calculated at the top of the spreadsheet: its result can be compared with the columns. The spreadsheet also provided the graphs in **Figure 2** and **Figure 3**.

Spreadsheets are superb for many problems in discrete mathematics. Knowing this, software companies have endowed current spreadsheets with hundreds of scientific functions. Just like formulas, the spreadsheet can be made to incorporate assumptions more realistic than those in the EOQ model. In many cases it is easier to do this with spreadsheets and more illuminating.

	Demand/unit time	order cost	holding cost item /unit time
	50	30	6
EOQ =	**22.3606797749979**		
Order Quantity	**Total Cost**	**Order Cost**	**Holding Cost**
1	1503.00	1500.00	3
2	756.00	750.00	6
3	509.00	500.00	9
4	387.00	375.00	12
5	315.00	300.00	15
6	268.00	250.00	18
7	235.29	214.29	21
8	211.50	187.50	24
9	193.67	166.67	27
10	180.00	150.00	30
11	169.36	136.36	33
12	161.00	125.00	36
13	154.38	115.38	39
14	149.14	107.14	42
15	145.00	100.00	45
16	141.75	93.75	48
17	139.24	88.24	51
18	137.33	83.33	54
19	135.95	78.95	57
20	135.00	75.00	60
21	134.43	71.43	63
22	134.18	68.18	66
23	134.22	65.22	69
24	134.50	62.50	72
25	135.00	60.00	75
26	135.69	57.69	78
27	136.56	55.56	81
28	137.57	53.57	84
29	138.72	51.72	87
30	140.00	50.00	90
31	141.39	48.39	93
32	142.88	46.88	96
33	144.45	45.45	99
34	146.12	44.12	102
35	147.86	42.86	105
36	149.67	41.67	108
37	151.54	40.54	111
38	153.47	39.47	114
39	155.46	38.46	117
40	157.50	37.50	120

Figure 4 A Spreadsheet Analysis of the Inventory Problem

Hadley, G. and T. M. Whitin. *Analysis of Inventory Systems*. 1963. Englewood Cliffs, New Jersey: Prentice-Hall.